

# Epi-two-dimensional flow and generalized enstrophy

Zensho Yoshida and Philip J. Morrison

**Abstract** The conservation of the enstrophy ( $L^2$  norm of the vorticity  $\omega$ ) plays an essential role in the physics and mathematics of two-dimensional (2D) Euler fluids. Generalizing to compressible ideal (inviscid and barotropic) fluids, the generalized enstrophy  $\int_{\Sigma(t)} f(\omega/\rho) \rho d^2x$  ( $f$  an arbitrary smooth function,  $\rho$  the density, and  $\Sigma(t)$  an arbitrary 2D domain co-moving with the fluid) is a constant of motion, and plays the same role. On the other hand, for the three-dimensional (3D) ideal fluid, the helicity  $\int_M \mathbf{V} \cdot \boldsymbol{\omega} d^3x$  ( $\mathbf{V}$  the flow velocity,  $\boldsymbol{\omega} = \nabla \times \mathbf{V}$ , and  $M$  the three-dimensional domain containing the fluid) is conserved. Evidently, the helicity degenerates in a 2D system, and the (generalized) enstrophy emerges as a compensating constant. This transition of the constants of motion is a reflection of an essential difference between 2D and 3D systems, because the conservation of the (generalized) enstrophy imposes stronger constraints, than the helicity, on the flow. In this paper, we make a deeper inquiry into the helicity-enstrophy interplay: the ideal fluid mechanics is cast into a Hamiltonian form in the phase space of Clebsch parameters, generalizing 2D to a wider category of epi-2D flows (2D embedded in 3D has zero helicity, while the converse is not true – our epi-2D category encompasses a wider class of zero-helicity flows); how helicity degenerates and is substituted by a new constant is delineated; and how a further generalized enstrophy is introduced as a constant of motion applying to epi-2D flow is described.

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Zensho Yoshida

Department of Advanced Energy, University of Tokyo, Chiba 277-8561, Japan, e-mail: yoshida@apl.k.u-tokyo.ac.jp

Philip J. Morrison

Department of Physics and Institute for Fusion Studies, University of Texas at Austin, TX 78712-1060, USA, e-mail: morrison@physics.utexas.edu

## 1 Introduction

The aim of this paper is to elucidate, from the perspective of Hamiltonian dynamics [1], how two-dimensional (2D) flow is different from general three-dimensional (3D) flow. Phenomenologically, 2D flow is often very different from 3D flow in that the former is less-turbulent and is more capable of generating and sustaining large-scale vortical structures – typhoons, jet streams, polar vortexes being spectacular examples of such structures created in atmospheric 2D flow. If we could delineate the root cause of such special behavior in 2D, we might be able to obtain a flow ‘intermediate’ between 2D and 3D, where the ‘regularity’ of 2D flow is maintained. As we will show, such is indeed possible.

We invoke the *helicity* as the key parameter for characterizing the transition from 3D to 2D (see Sec. 2). As is well known, the helicity is a constant of motion in an ‘ideal’ flow (in this paper ideal means inviscid and barotropic). In 2D geometry, however, the helicity degenerates to zero; but as a compensation, the *enstrophy* (or its generalization, cf. Remark 1) becomes a nontrivial constant (see Secs. 2.2 and 3.3). The conservation of the (generalized) enstrophy is a most essential property for distinguishing 2D from 3D. The enstrophy is a higher-order functional in comparison with the helicity, and its conservation is deemed to be reason for the aforementioned difference between 2D and 3D systems. Even when the constancy of the enstrophy or the helicity is broken by the inclusion of dissipation, the macroscopic structure of the fluid system is strongly influenced by these ideal constants of motion (cf. [2]).

Needless to say, zero-helicity flow is not necessarily 2D. In Sec. 4, we introduce our category of ‘epi-2D’ flow that maintains the basic properties of zero-helicity and enstrophy conservation, while not necessarily being 2D. Having cast ideal fluid mechanics into a Hamiltonian form in the phase space of Clebsch parameters (e.g. [3, 4, 5, 6, 7]), the category of epi-2D flow is, then, defined as a reduction of the phase space. One of the reduced parameter used is a *phantom* [8, 9], by which we define a generalized enstrophy. In Sec. 5 we introduce the notion of an epi-2D particle to elucidate our theory.

## 2 Preliminaries

### 2.1 Three-dimensional fluid mechanics

We start by reviewing the basic equations of fluid mechanics and the associated conservation laws. Here we use the conventional notation of 3D vector analysis, with vector fields denoted by bold-face symbols.

Let  $M$  be a 3D domain containing an ideal fluid. For simplicity, we assume  $M = T^3$ , the 3-torus, and ignore the effect of boundaries. We denote by  $\rho$  the mass density,  $\mathbf{V}$  the fluid velocity, and  $P$  the pressure. Thus, the governing equations are

$$\partial_t \rho = -\nabla \cdot (\mathbf{V} \rho), \quad (1)$$

$$\partial_t \mathbf{V} = -(\mathbf{V} \cdot \nabla) \mathbf{V} - \rho^{-1} \nabla P. \quad (2)$$

Assuming a barotropic relation  $P = P(\rho)$ , with  $\rho^{-1} \nabla P = \nabla h$  for  $h = h(\rho)$ , the energy of the system is

$$H = \int_M \left[ \frac{1}{2} |\mathbf{V}|^2 + \varepsilon(\rho) \right] \rho \, d^3x, \quad (3)$$

where  $\varepsilon(\rho)$  is the specific thermal energy, satisfying  $\partial(\rho \varepsilon(\rho))/\partial \rho = h(\rho)$ . Evidently,  $dH/dt = 0$ .

The vorticity  $\boldsymbol{\omega} = \nabla \times \mathbf{V}$  obeys the vorticity equation, obtained by taking the curl of (2), i.e.,

$$\partial_t \boldsymbol{\omega} = \nabla \times (\mathbf{V} \times \boldsymbol{\omega}). \quad (4)$$

Evidently, the total mass  $N = \int_M \rho \, d^3x$  is a constant of motion, along with another conserved quantity, the helicity:

$$C = \int_M \mathbf{V} \cdot \boldsymbol{\omega} \, d^3x. \quad (5)$$

Using (2) and (4), we easily verify  $dC/dt = 0$ .<sup>1</sup>

## 2.2 Two-dimensional fluid mechanics

To compare 2D and 3D systems, it is convenient to immerse a 2D system into 3D space. For simplicity, we consider a flat torus  $T^2$ , on which we define Cartesian coordinates  $x$  and  $y$ . We add a ‘perpendicular’ coordinate  $z$  and extend  $T^2$  to  $T^3$ , with  $\mathbf{e}_z = \nabla z$ , which we call the perpendicular vector. Now we may define a 2D system by the reduction of the 3D system with  $\mathbf{e}_z \cdot = 0$  and  $\partial_z = 0$ . Indeed, a 2D fluid model is formulated by such a reduction.<sup>2</sup> We interpret a 2D flow  $\mathbf{v} = (v_x, v_y)^T$  as a special 3D flow such that  $\mathbf{V} = (v_x, v_y, 0)^T$ . The vorticity can be defined as  $\boldsymbol{\omega} = \nabla \times \mathbf{V} = \omega \mathbf{e}_z$  with  $\omega = \partial_x v_y - \partial_y v_x$ . In which case the vorticity equation (4) reduces to a single-component equation:

$$\partial_t \omega = -\nabla \cdot (\mathbf{v} \omega). \quad (6)$$

<sup>1</sup> In this work, we do not argue for the the existence of regular solutions of the model equations. The conservation laws discussed are, therefore, *a priori* relations satisfied by all regular solutions if they exist.

<sup>2</sup> We note that  $\partial_z = 0$  does not mean that the system extends in the  $z$ -direction homogeneously; instead, we consider a thin system in which variation of physical quantities in the  $z$ -direction is much larger than in the  $x$  and  $y$  directions. Thus,  $\partial_z$  can be separated from  $\partial_x$  and  $\partial_y$ .

Because  $\mathbf{V} \cdot \boldsymbol{\omega} = 0$ , the helicity conservation is now trivial,  $C \equiv 0$ ; however, interestingly, a new constant emerges that replaces the degenerated helicity.<sup>3</sup> By (6) and the mass conservation law, which now reads  $\partial_t \rho = -\nabla \cdot (\mathbf{v}\rho)$ , we obtain, the following equation for the *potential vorticity*,  $\vartheta = \omega/\rho$ :

$$\partial_t \vartheta = -\mathbf{v} \cdot \nabla \vartheta, \quad (7)$$

and we define the *generalized enstrophy* by

$$Q = \int_M f(\vartheta) \rho \, d^2x, \quad (8)$$

where  $f$  is an arbitrary  $C^1$ -class function. Using (7) and the mass conservation law, we can easily verify that  $dQ/dt = 0$ .

*Remark 1.* For an incompressible flow ( $\nabla \cdot \mathbf{v} = 0$ ), we may assume  $\rho = \text{constant}$ , and then, (8) has a special form of  $\int_M \omega^2 d^2x$ , which is the usual *enstrophy*.

We end this introductory section with drawing attention to the fact that all constants of motion, i.e. the total mass  $N$ , the helicity  $C$ , and the generalized enstrophy  $Q$  are defined by the spatial integrals over the 3D or 2D domain. This means that the integrand of a constant of motion defines an  $n$ -form ( $n$  the spatial dimension) in the language of differential geometry. In the following analysis, this fact guides our formulation of generalized enstrophy.

### 3 Topological invariants in fluid motion

#### 3.1 Hamiltonian formalism of ideal fluid motion

For the study of geometrical properties of fluid mechanics, we reformulate the governing equations in the framework of differential geometry. We first introduce a *phase space*  $X$  that hosts the underlying state vectors  $\boldsymbol{\xi}$ ; the physical quantity  $\mathbf{u} = (\rho, \mathbf{V})^T$  ( $\in \mathcal{V}$ , the space of physical variables) is some function parameterized by  $\boldsymbol{\xi}$ . Let

$$\boldsymbol{\xi} = (\varphi, \varrho, q, p, r, s)^T \in X, \quad (9)$$

where  $\xi_1 = \varphi, \xi_3 = q, \xi_5 = r$  are 0-forms and  $\xi_2 = \varrho, \xi_4 = p, \xi_6 = s$  are  $n$ -forms in the base space  $M = T^3$ . We assume  $\xi_j$  ( $j = 1, \dots, 6$ ) are smooth (i.e.  $C^\infty$ -class) functions. The dual space  $X^*$  is the Hodge-dual of  $X$ , i.e. the odd number components of  $\boldsymbol{\eta} \in X^*$  are  $n$ -forms and the even number components are 0-forms. The pairing of  $X^*$  and  $X$  is <sup>4</sup>

<sup>3</sup> Fukumoto [10] points out that the helicity and the generalized enstrophy can be unified by the concept of *cross helicity*.

<sup>4</sup> Here the phase space (function space)  $X$  may be viewed as a cotangent bundle of  $X_q = \{(\xi_1, \xi_3, \xi_5)^T; \xi_j \in \Omega^0(M)\}$ . For  $F \in C^\infty(X)$ ,  $\partial_{\boldsymbol{\xi}} F \in X^*$  (to be defined in (12)) may be regarded as

$$\langle \boldsymbol{\eta}, \boldsymbol{\xi} \rangle = \sum_j \int_M \eta_j \wedge \xi_j, \quad \boldsymbol{\eta} \in X^*, \boldsymbol{\xi} \in X. \quad (10)$$

On the space  $C^\infty(X)$  of *observables*, we define a canonical Poisson bracket

$$\{F, G\} = \langle \partial_{\boldsymbol{\xi}} F, J \partial_{\boldsymbol{\xi}} G \rangle, \quad (11)$$

where  $F, G \in C^\infty(X)$ ,  $\partial_{\boldsymbol{\xi}} F$  is the gradient of  $F$  defined by

$$F(\boldsymbol{\xi} + \varepsilon \boldsymbol{\zeta}) - F(\boldsymbol{\xi}) = \varepsilon \langle \partial_{\boldsymbol{\xi}} F, \boldsymbol{\zeta} \rangle + O(\varepsilon^2) \quad (\forall \boldsymbol{\zeta} \in X), \quad (12)$$

and  $J : X^* \rightarrow X$  is the symplectic operator

$$J = J_c \oplus J_c \oplus J_c, \quad J_c = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \quad (13)$$

We denote by  $C_{\{\cdot, \cdot\}}^\infty(X)$  the Poisson algebra of observables on  $X$ . The adjoint representation of Hamiltonian dynamics is, for a given Hamiltonian  $H$ ,

$$\frac{d}{dt} F = \{F, H\}, \quad (14)$$

which is equivalent to Hamilton's equation of motion

$$\frac{d}{dt} \boldsymbol{\xi} = J \partial_{\boldsymbol{\xi}} H. \quad (15)$$

We relate the physical quantity  $\boldsymbol{u} \in \mathcal{V}$  and  $\boldsymbol{\xi} \in X$  by  $\boldsymbol{\rho} \Leftrightarrow \varrho^*$  (i.e.  $\varrho^* \text{vol}^n = \boldsymbol{\rho}$  with the volume  $n$ -form  $\text{vol}^n$ ; here  $n = 3$ ),<sup>5</sup> and

$$\boldsymbol{V} \Leftrightarrow \wp = d\varphi + \check{p}dq + \check{s}dr, \quad (\check{p} = p^*/\varrho^*, \check{s} = s^*/\varrho^*). \quad (16)$$

Writing a vector as (16) is called the *Clebsch parameterization*. The five Clebsch parameters  $(\varphi, q, \check{p}, r, \check{s})$  are sufficient to represent every 3-vector (1-form in 3D space) [7]. Inserting (16) into the fluid energy (3), we obtain a Hamiltonian

$$H(\boldsymbol{\xi}) = \int_M \left[ \frac{1}{2} |d\varphi + (p^*/\varrho^*)dq + (s^*/\varrho^*)dr|^2 + \varepsilon(\varrho^*) \right] \varrho. \quad (17)$$

With this  $H$ , the equation of motion (15) reads

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a '1-form' on  $X$ . Hence, the duality of  $X^*$  and  $X$  corresponds to that of 'co-vectors' and 'vectors'. At the same time, the components of the field  $\boldsymbol{\xi} \in X$  are differential forms (0-forms and  $n$ -forms) on the 'base space'  $M$ ; the 'Hodge-duality' of  $X^*$  and  $X$  is in the sense of the differential forms on  $M$ , while the duality (10) is in the sense of 'co-vectors' and 'vectors' on the function space  $X$ .

<sup>5</sup> Here we denote by  $\alpha^*$  the Hodge dual of a differential form  $\alpha$ .

$$\widetilde{\mathcal{L}}_{\mathbf{V}}\varphi = h - V^2/2, \quad (18)$$

$$\widetilde{\mathcal{L}}_{\mathbf{V}}q = 0, \quad \widetilde{\mathcal{L}}_{\mathbf{V}}r = 0, \quad (19)$$

$$\widetilde{\mathcal{L}}_{\mathbf{V}}\varrho = 0, \quad \widetilde{\mathcal{L}}_{\mathbf{V}}p = 0, \quad \widetilde{\mathcal{L}}_{\mathbf{V}}s = 0, \quad (20)$$

where we denote

$$\widetilde{\mathcal{L}}_{\mathbf{V}} = \partial_t + \mathcal{L}_{\mathbf{V}}, \quad (21)$$

with  $\mathcal{L}_{\mathbf{V}}$  being the conventional Lie derivative along the vector  $\mathbf{V} \in TM$ .<sup>6 7</sup>

The first equation of (20) is nothing but the mass conservation law (1). Evaluating  $\partial_t \mathbf{V}$  by inserting (16) and using (18)–(20), we obtain (2). Hence, Hamilton's equation (15) with the Hamiltonian (17) describes the fluid motion obeying (1) and (2).<sup>8</sup>

### 3.2 Gauge symmetry and helicity

In (16), the Clebsch parameters are apparently a redundant representation of a 3-vector  $\mathbf{V}$ . In fact, the map  $X \rightarrow \mathcal{V}$  is not an injection (although a surjection) [7]. For example, the transformation

$$\varphi \mapsto \varphi + \varepsilon \quad (\varepsilon \in \mathbb{R}) \quad (22)$$

does not change the physical quantity  $\mathbf{u} \in \mathcal{V}$ . Such a map is called a *gauge transformation*. We find that the map (22) is a Hamiltonian flow generated by the constant of motion  $N = \int_M \varrho$ , i.e. the map (22) may be written as  $(I + \varepsilon J \partial_{\xi} N)$ . Or, the co-adjoint orbit  $\text{Ad}_N^*(\varepsilon)$  is a gauge-transformation group of the Clebsch parameterization.

The helicity  $C$ , which now reads

$$C = \int_M \vartheta \wedge d\vartheta, \quad (23)$$

yields a different gauge group  $\text{Ad}_C^*(\varepsilon)$  (see [13] for the explicit form the corresponding gauge transformation).

*Remark 2.* If we denote by  $\{F, G\}_{\mathcal{V}}$  the Poisson bracket of (11) evaluated only for observables  $F$  and  $G$  in which  $\xi$  appears in terms of the Clebsch parameterized  $\mathbf{u} \in \mathcal{V}$ , we obtain

$$\{G, N\}_{\mathcal{V}} = 0, \quad \{G, C\}_{\mathcal{V}} = 0 \quad \forall G. \quad (24)$$

<sup>6</sup> Here  $\mathbf{V}$  is regarded as a vector  $\in TM$  through the following identification. By (16),  $\mathbf{V} \Leftrightarrow \vartheta \in T^*M$ . In the Hamiltonian (17),  $|\vartheta|^2 \Leftrightarrow \mathbf{V}^\dagger \cdot \mathbf{V}$  with the dual  $\mathbf{V}^\dagger = \mathbf{V} \in TM$ .

<sup>7</sup> When considering a relativistic fluid, we generate a diffeomorphism group  $e^{\tau U}$  ( $\tau$  the proper time), acting on a space-time manifold  $\tilde{M} = \mathbb{R} \times M$ , by a space-time velocity  $U \in T\tilde{M}$ . Then, the space-time derivative  $\widetilde{\mathcal{L}}_{\mathbf{V}}$  is replaced by the natural Lie derivative  $\mathcal{L}_U$ . When  $\mathcal{L}_U$  applies to a differential form  $\alpha$  on  $\tilde{M}$ , the temporal and spatial components are mixed up (cf. [11]).

<sup>8</sup> See [12] for the underlying action principle that yields the canonical system of Hamilton's equation (18)–(20).

Hence,  $N$  and  $C$  are the Casimir elements of the reduced Poisson algebra  $C_{\{\cdot, \cdot\}_{\mathcal{V}}}^{\infty}(\mathcal{V})$ . See e.g. [14] for general discussion on *reduction* of Poisson brackets. The existence of Casimir elements is the characteristics of *noncanonical Poisson brackets* [1, 15]. We note that  $C_{\{\cdot, \cdot\}_{\mathcal{V}}}^{\infty}(\mathcal{V})$  has much more (in fact, infinitely many) topological constraints (constants of motions) that are not integrable, i.e., do not define Casimir elements (see [8]).

### 3.3 Two-dimensional system and generalized enstrophy

In a 2D system ( $M = T^2$ ), we can parameterize a general 2D velocity as

$$\mathbf{V} \Leftrightarrow \wp = d\varphi + \check{p}dq. \quad (25)$$

Now only three Clebsch parameters  $\varphi$ ,  $\check{p}$ , and  $q$  suffice [7]. Hence, the phase space is

$$Z = \{\boldsymbol{\zeta} = (\varphi, \varrho, q, p)^T; \varphi, q \in \Omega^0(T^2), \varrho, p \in \Omega^2(T^2)\}. \quad (26)$$

All other formalisms are the same as the case of 3D systems. However, because the 3-form  $\wp \wedge d\wp$  cannot be defined in 2D space we do not have the helicity conservation law.

As mentioned in Sec. 2.2, a different constant of motion emerges in 2D, the generalized enstrophy, which is a spatial (2D) integral of a 2-form that involves the vorticity  $\omega = d\wp$ . As a preparation for the development in the next section, we reformulate the generalized enstrophy in the language of differential geometry (with a slight extension), and re-prove its conservation. Let

$$Q = \int_{\Sigma(t)} f(\omega^* / \varrho^*) \varrho, \quad (27)$$

where  $f$  is an arbitrary smooth function, and  $\Sigma(t) \subset M$  is a co-moving ‘volume’ (in fact, a 2D surface). Notice that the integral is evaluated on a subset  $\Sigma(t)$  that is moved by the group-action of  $e^{t\mathbf{V}}$ .

By the following Lemma 1 and the mass conservation law  $\widetilde{\mathcal{L}}_{\mathbf{V}} \varrho = 0$ , we find

$$\begin{aligned} \frac{d}{dt} Q &= \int_{\Sigma(t)} \widetilde{\mathcal{L}}_{\mathbf{V}} [f(\omega^* / \varrho^*) \varrho] \\ &= \int_{\Sigma(t)} \left[ f' \varrho \widetilde{\mathcal{L}}_{\mathbf{V}} (\omega^* / \varrho^*) + f \widetilde{\mathcal{L}}_{\mathbf{V}} \varrho \right] = 0. \end{aligned} \quad (28)$$

**Lemma 1.** *Let  $\alpha$  and  $\beta$  be a pair of  $n$ -forms defined on a smooth manifold  $M$  of dimension  $n$ . Denoting  $\alpha = \alpha^* \text{vol}^n$  and  $\beta = \beta^* \text{vol}^n$ , we define  $\vartheta = \alpha^* / \beta^*$ . If  $\widetilde{\mathcal{L}}_{\mathbf{V}} \alpha = 0$  and  $\widetilde{\mathcal{L}}_{\mathbf{V}} \beta = 0$  for a vector  $\mathbf{V} \in TM$ , then  $\widetilde{\mathcal{L}}_{\mathbf{V}} \vartheta = 0$ .*

*Proof.* By the definition,

$$\widetilde{\mathcal{L}}_{\mathbf{V}}\alpha = (\widetilde{\mathcal{L}}_{\mathbf{V}}\alpha^*)\text{vol}^n + \alpha^*(\widetilde{\mathcal{L}}_{\mathbf{V}}\text{vol}^n) = (\widetilde{\mathcal{L}}_{\mathbf{V}}\alpha^*)\text{vol}^n + \alpha^*(\text{div } \mathbf{V})\text{vol}^n.$$

When  $\widetilde{\mathcal{L}}_{\mathbf{V}}\alpha = 0$ , we may write  $\widetilde{\mathcal{L}}_{\mathbf{V}}\alpha^* = -\alpha^*\text{div } \mathbf{V}$ . The same formula applies to  $\widetilde{\mathcal{L}}_{\mathbf{V}}\beta^*$ . We thus have

$$\widetilde{\mathcal{L}}_{\mathbf{V}}\left(\frac{\alpha^*}{\beta^*}\right) = \frac{\widetilde{\mathcal{L}}_{\mathbf{V}}\alpha^*}{\beta^*} - \frac{\alpha^*\widetilde{\mathcal{L}}_{\mathbf{V}}\beta^*}{\beta^{*2}} = \frac{-\alpha^*\text{div } \mathbf{V}}{\beta^*} + \frac{\alpha^*\beta^*\text{div } \mathbf{V}}{\beta^{*2}} = 0.$$

□

We want to generalize  $Q$  to a class of 3D systems by considering a 3-form integral of the form

$$Q = \int_{V(t)} f(\vartheta) \varrho, \quad (29)$$

for some scalar  $\vartheta$  that reflects  $\omega$ .

## 4 Epi-two dimensional flow

### 4.1 Reduction of the phase space

A thought, drawn from the foregoing observation, is that the degeneration of one constant of motion (i.e. the helicity) must be compensated by a new constant of motion (i.e. the generalized enstrophy). Although we have seen that the degeneracy of the helicity is usual for 2D systems, it may occur in a more general situation. Then, it is conceivable that the compensation should also occur simultaneously. If so, a generalized enstrophy may exist as a topological constraint in a wider class of ideal flows, which we call *epi-2D flows*.

**Definition 1 (epi-2D flow).** Let  $Y$  be a phase space of smooth Clebsch parameters such that

$$Y = \{\boldsymbol{\eta} = (\varphi, \varrho, q, p)^T; \varphi, q \in \Omega^0(T^3), \varrho, p \in \Omega^3(T^3)\}. \quad (30)$$

The corresponding physical fields  $\rho \Leftrightarrow \varrho^*$  and

$$\mathbf{V} \Leftrightarrow \wp = d\varphi + \left(\frac{p^*}{\varrho^*}\right) dq \quad (31)$$

are called epi-two-dimensional (epi-2D) flows.

Notice the difference between  $Y$  of (30) and  $Z$  of (26); in particular, the epi-2D flows are defined on the 3D domain  $T^3$ . The reduced phase space  $Y$  is a closed subset of  $X$ . We denote by  $\langle \cdot, \cdot \rangle_Y$  the reduced pairing of  $Y^*$  and  $Y$  (cf. (10)). By restricting observables in  $Y$ , we define a canonical Poisson algebra  $C_{\{\cdot, \cdot\}_Y}^\infty(Y)$ , where the Poisson bracket is



$$\{F, G\}_Y = \langle \partial_{\boldsymbol{\eta}} F, J_Y \partial_{\boldsymbol{\eta}} G \rangle_Y, \quad J_Y = J_c \oplus J_c. \quad (32)$$

Epi-2D flow may have a finite vorticity  $d\wp = d\check{p} \wedge dq$ , where  $\check{p} = p^* / \varrho^*$ . However, we observe

$$\wp \wedge d\wp = d\wp \wedge d\check{p} \wedge dq = d(\wp \wedge d\check{p} \wedge dq),$$

i.e. the helicity density  $\wp \wedge d\wp$  is an exact 3-form. Hence, we have

**Proposition 1.** *Epi-2D flow has zero helicity, i.e.  $C = \int_M \wp \wedge d\wp = 0$ .*

A *vortex line* is a curve determined by

$$\frac{d}{d\tau} \mathbf{x} = \boldsymbol{\omega}(\mathbf{x}), \quad (33)$$

where  $\boldsymbol{\omega}$  is the vorticity. For epi-2D flow,  $\boldsymbol{\omega} = \nabla \check{p} \times \nabla q$  ( $\Leftrightarrow d\wp = d\check{p} \wedge dq$ ). Evidently, the level-sets of  $\check{p}$  and  $q$  are the ‘integral surfaces’ of vortex lines:

$$\begin{aligned} \frac{d}{d\tau} \check{p}(\mathbf{x}(\tau)) &= \nabla \check{p} \cdot \frac{d}{d\tau} \mathbf{x} = \nabla \check{p} \cdot \boldsymbol{\omega}(\mathbf{x}) = 0, \\ \frac{d}{d\tau} q(\mathbf{x}(\tau)) &= \nabla q \cdot \frac{d}{d\tau} \mathbf{x} = \nabla q \cdot \boldsymbol{\omega}(\mathbf{x}) = 0. \end{aligned}$$

This well-known fact can be stated as

**Proposition 2.** *The vortex line equation of epi-2D flow is integrable; two Clebsch parameters  $\check{p}$  and  $q$  define the integral surfaces. We call the surface spanned by  $d\check{p}$  and  $dq$  the vortex surface.*

The epi-2D flow generated by a reduced Hamiltonian

$$H(\boldsymbol{\eta}) = \int_M \left[ \frac{1}{2} |d\wp + (p^* / \varrho^*) dq|^2 + \varepsilon(\varrho^*) \right] \varrho \quad (34)$$

satisfies the 3D fluid equations (1) and (2).

We may observe the epi-2D dynamics in the larger phase space  $X$ . Since the reduced Hamiltonian  $H(\boldsymbol{\eta})$  does not include the variables  $r$  and  $s$ , the flow velocity  $\mathbf{V}$  is independent of  $r$  and  $s$ . However, they obey the same equations (19) and (20), i.e.

$$\widetilde{\mathcal{L}}_{\mathbf{V}} r = 0, \quad \widetilde{\mathcal{L}}_{\mathbf{V}} s = 0. \quad (35)$$

Such fields, co-moving with the epi-2D flow, are called *phantoms* [8, 9]. Every functional such as  $F(r, s)$  is a constant of motion:  $\{F(r, s), H(\boldsymbol{\eta})\} = 0$ .

## 4.2 Generalized enstrophy of epi-2D flow

In light of the above, it is not surprising that we have a family of conservation laws for epi-2D fluid motion:

**Theorem 1.** *Let  $\boldsymbol{\eta}(t)$  be an epi-2D flow generated by the reduced Hamiltonian  $H(\boldsymbol{\eta})$  of (34), and  $r(t)$  be a co-moving phantom. We define a generalized enstrophy (denoting  $\omega = d\check{p} \wedge dq$ )*

$$Q = \int_{\Omega(t)} f(\vartheta) \varrho, \quad \vartheta = \frac{(\omega \wedge dr)^*}{\varrho^*} \quad (36)$$

with an arbitrary smooth function  $f$  and an arbitrary co-moving 3D volume element  $\Omega(t) \subset M$ . Then,  $dQ/dt = 0$ .

*Proof.* Since  $\widetilde{\mathcal{L}}_{\mathbf{V}} \varrho = 0$ , what we have to prove is  $\widetilde{\mathcal{L}}_{\mathbf{V}} \vartheta = 0$ . We have

$$\begin{aligned} \widetilde{\mathcal{L}}_{\mathbf{V}}(d\check{p} \wedge dq \wedge dr) &= (\widetilde{\mathcal{L}}_{\mathbf{V}} \omega) \wedge dr + \omega \wedge \widetilde{\mathcal{L}}_{\mathbf{V}} dr \\ &= \omega \wedge d(\widetilde{\mathcal{L}}_{\mathbf{V}} r) = 0. \end{aligned}$$

By Lemma 1, we obtain  $\widetilde{\mathcal{L}}_{\mathbf{V}} \vartheta = 0$ .  $\square$

The generalized enstrophy (36) is a 3-dimensional generalization of the two-dimensional one (27). About its application, we have the following remarks:

*Remark 3.* In a general 3D flow,  $Q$  is also a constant of motion. However, it does not characterize the vorticity, since  $\omega$  must be inflated to  $d\wp = d\check{p} \wedge dq + d\check{s} \wedge dr$ .

*Remark 4.* In the case of 2D flow, we may first immerse the system into 3D by adding a perpendicular coordinate  $z$  (see Sec. 2.2), and take the phantom  $r = z$  (this  $r$  is stationary). Then, Theorem 1 reproduces the result of Sec. 3.3.

*Remark 5.* Suppose that  $\omega \neq 0$ . Then, we may choose the initial value of  $r$  so that

$$(d\check{p} \wedge dq \wedge dr)^* = \frac{D(\check{p}, q, r)}{D(x, y, z)} \neq 0. \quad (37)$$

Hence, the generalized enstrophy can be made nontrivial. Analogous to the 2D generalized enstrophy, such an  $r$  is a coordinate co-moving with the fluid, which penetrates the vortex surface (see Proposition 2).

## 5 A particle picture

### 5.1 Epi-2D ‘particles’

We can exploit *local* epi-2D regions in order to define particle-like behavior. In the general 3D parameterization  $\mathbf{V} \Leftrightarrow \wp = d\phi + \check{p}dq + \check{s}dr$ , a region in which  $\check{s} = 0$

may be called a *epi-2D domain*. Since  $\check{s}$  co-moves with the fluid, every infinitesimal volume element (denoted by  $\Omega_j(t)$  with  $j$  an index for each such volume element) included in a *epi-2D domain* may be viewed as a quasiparticle, which we call an *epi-2D particle*. The generalized enstrophy evaluated for  $\Omega_j(t)$ , which we denote by  $Q_1(\Omega_j)$  is a constant of motion, characterizing the vorticity included there. We call  $Q_1(\Omega_j)$  the *charge* of the epi-2D particle  $\Omega_j$ .

A symmetric epi-2D particle can be defined by a domain in which  $\check{p} = 0$ , with the corresponding generalized enstrophy given by

$$Q_2(\Omega_j) = \int_{\Omega_j(t)} f(\vartheta_2) \varrho, \quad \vartheta_2 = \frac{(\omega_2 \wedge dq)^*}{\varrho^*},$$

measuring the vorticity  $\omega_2 = d\check{s} \wedge dr$ .

As noted in Remark 3, both  $Q_1$  and  $Q_2$  can be evaluated in a general co-moving domain (particle)  $\Omega(t) \subset M$ , and they are ubiquitous constants. However, they do not represent the ‘enstrophy’ of an actual vorticity when the vorticity exists in a mixed state  $d\check{p} \wedge dq + d\check{s} \wedge dr$ . Hence, we may interpret  $Q_1$  and  $Q_2$  as ‘potential’ quantities, which become ‘observable’ when one of  $\omega_1 = d\check{p} \wedge dq$  or  $\omega_2 = d\check{s} \wedge dr$  alone occupies a domain.

## 5.2 Discovering epi-2D particle

In the preceding subsection, the notion of an epi-2D particle (or domain) was introduced using the Clebsch parameters which are the potential fields lying beneath the observables. Here we make an attempt to discover an epi-2D particle only from the physical variable  $\mathbf{u}$ .

We start by remembering the well-known relation:

**Lemma 2 (Frobenius).** *Let  $\wp$  be a  $C^1$ -class 1-form on a smooth manifold  $M$  of dimension  $n$  ( $\geq 3$ ). The following two conditions are equivalent:*

1.  $\wp$  has zero helicity density, i.e.,

$$\wp \wedge d\wp = 0. \quad (38)$$

2.  $\wp$  is locally (i.e., in a neighborhood  $\Omega$  of every point of  $M$ ) integrable, i.e., there exist two scalars  $\alpha$  and  $\beta$  by which  $\wp$  can be written as

$$\wp = \alpha d\beta. \quad (39)$$

Then, the Pfaffian equation  $\wp = 0$  foliates  $\Omega$  by the level-sets of  $\beta$ .

We define a quotient space

$$V_s = \Omega^1(M)/d\Omega^0(M), \quad (40)$$

which may be identified as the space of *solenoidal vector fields*. If we identify a 1-form  $\mathcal{J} \in V_s$  with a 3-vector field  $\mathbf{V}$ , the integral  $\int_M \mathcal{J} \wedge d\mathcal{J}$  evaluates the helicity  $C = \int_M \mathbf{V} \cdot \nabla \times \mathbf{V} d^3x$ . Notice that the transformation  $\mathbf{V} \mapsto \mathbf{V} + \nabla\phi$  ( $\forall\phi$ ) does not change the helicity, while the integrand (helicity density)  $\mathbf{V} \cdot \nabla \times \mathbf{V}$  is modified. But by defining the helicity density  $\mathcal{J} \wedge d\mathcal{J}$  on  $V_s$ , the gauge ( $d\phi$ ) dependence has been removed.

If  $\mathcal{J} \wedge d\mathcal{J} = 0$  in  $\Omega \subset M$ , we say that  $\mathcal{J}$  is ‘helicity free’ in  $\Omega$ . By Lemma 2, a helicity-free  $\mathcal{J}$  can be represented as  $\mathcal{J} = \check{p}dq$  ( $\exists\check{p}, q$ ) in some  $\Omega' \subset \Omega$ , which implies that  $\mathcal{J}' = \mathcal{J} + d\phi$  ( $\exists\phi$ ) can be identified with the flow velocity  $\mathbf{V}$  in  $\Omega'$ . To put it in another way, we have

**Proposition 3.** *Given that the projection onto  $V_s$  of a flow velocity  $\mathbf{V}$  is helicity free in  $\Omega \subset M$ , i.e., there exists  $\nabla\phi$  by which we can make  $\mathbf{V} - \nabla\phi \sim \mathcal{J} \in V_s$  such that  $\mathcal{J} \wedge d\mathcal{J} = 0$  in  $\Omega$ , then such a  $\mathbf{V}$  is epi-2D in some  $\Omega' \subset \Omega$ .*

## 6 Conclusion

Diverse structures generated in fluids are not attributed to features of some nontrivial energy functional. In fact, the energy of a usual fluid is quite simple, it being the equivalent a *norm* on phase space of physical variables such as that given by (3). This is in marked contrast to the usual situation in condensed-matter physics where, for example, phase transitions or spinodal decompositions are modeled by bumpy energies. The key role for fluids is, then, played by ‘constraints’ that forbid the dynamics from obeying simple orbits that might be determined by the energy alone. In the ideal (no-dissipation) limit, such constraints are manifested as conservation laws. Indeed, ideal fluid mechanics has infinitely many such constants of motion, and some of them are essential for controlling bifurcations of diverse structures or maintaining stability of some vortical motion. In the present work, we focused on two well-known constants of motion: the helicity of 3D flow and the (generalized) enstrophy of 2D flow, and we studied the basic mechanism of their creation.

In physics, a constant of motion is expected to be the product of some *symmetry*. However, the energy (Hamiltonian) of the fluid, represented in terms of the usual physical (Eulerian) variables, does not bear such symmetries to produce the helicity or enstrophy. Therefore, we are led to consider a set of underlying basic parameters beneath the physical quantities, and assume that some specific combinations of them appear as observables. Here, we invoked Clebsch parameters, and showed that the helicity is the product of gauge symmetry of the Clebsch parameterization.

We have observed that the phase space  $X$  of general 3D flows is hierarchically foliated into submanifolds, where the smallest subsystem hosts vorticity-free (irrotational) flows. The next hierarchy  $Y$  hosts the epi-2D flows, which is a subset of the zero-helicity leaf (Proposition 1). The subsystem  $Y$  is foliated by the generalized enstrophy (Theorem 1), which is a 3D generalization of the conventional one for 2D systems. Notice that the reduction from  $X$  (general 3D flow) to  $Y$  (epi-2D flow) is

not a geometrical constraint (cf. the 2D system of Sec. 3.3). However, epi-2D systems have intrinsic vortex surfaces (Proposition 2), which parallels the *a priori* base space of the 2D system. The generalized enstrophy is the measure of the circulation on such vortex surfaces.

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